# Chapter 17: Binomial option pricing

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The general case

# ONE-PERIOD OPTION PRICING

# One-period binomial: State Prices

- **Stock** with price $S_0$ today and price:
  - $S_{Up} = S_0 * U$ in <u>up state</u> and
  - $S_{Down} = S_0 * D$ in <u>down state</u>
- U and D are one-plus-return on the stock.
  - For example, U=1+10%, D=1-3%.
- U and D stand for <u>one-plus</u> the return

# One-period binomial, numerical example

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 2 | Up, U | 1.10 | | | |
| 3 | Down, D | 0.97 | | | |
| 4 | | | | | |
| 5 | Initial stock price | 50.00 | | | |
| 6 | Interest rate, R | 1.06 | | | |
| 7 | Exercise price | 50.00 | | | |
| 8 | | | | | |
| 9 | | **Stock price** | | | |
| 10 | | | | 55.00 | <-- =$B$11*B2 |
| 11 | | 50.00 | | | |
| 12 | | | | 48.50 | <-- =$B$11*B3 |
| 13 | | | | | |
| 14 | | **Call option** | | | |
| 15 | | | | 5.00 | <-- =MAX(D10-$B$7,0) |
| 16 | | ??? | | | |
| 17 | | | | 0.00 | <-- =MAX(D12-$B$7,0) |

**In this example: $S_0$ = 50, U (up move) = 1.10, D (down move) = 0.97, and R (1+interest) = 1.06.**

**The call option has exercise price of 50.**

# State prices: Alternative interpretation of binomial model

- Denote by $q_U$ the price today of $1 in the <u>up state</u> at date 1
- Denote by $q_D$ the price today of $1 in the <u>down state</u> at date 1
- The two state prices should price all assets including stock and bond:

$$Stock: \underbrace{55q_U}_{\substack{\text{PV of stock} \\ \text{payoff in Up} \\ \text{state}}} + \underbrace{48.5q_D}_{\substack{\text{PV of stock} \\ \text{payoff in Down} \\ \text{state}}} = \underbrace{50}_{\substack{\text{Stock price} \\ \text{today}}}$$

# Solving for $q_U$ and $q_D$ using stock prices

$$50 = 55q_U + 48.5q_D$$

$$q_U = \frac{R - D}{R*(U - D)} = \frac{1.06 - 0.97}{1.06*(1.10 - 0.97)} = 0.6531$$

$$q_D = \frac{U - R}{R*(U - D)} = \frac{1.1 - 1.06}{1.06*(1.10 - 0.97)} = 0.2903$$

Those are not the risk-neutral probability!

# State prices depend only on U, D, R
# Not on stock price $S_0$

$$50 = 55q_U + 48.5q_D = 50 * U * q_U + 50 * D * q_D$$

$or:$

$$1 = \underbrace{1.10\, q_U}_{\bar{U}} + \underbrace{0.97\, q_D}_{\bar{D}}$$

$$q_U = \frac{R - D}{R * (U - D)} = \frac{1.06 - 0.97}{1.06 * (1.10 - 0.97)} = 0.6531$$

$$q_D = \frac{U - R}{R * (U - D)} = \frac{1.1 - 1.06}{1.06 * (1.10 - 0.97)} = 0.2903$$

# In Excel (note check in rows 11/12)

| | A | B | C |
|---|---|---|---|
| 1 | **DERIVING THE STATE PRICES** | | |
| 2 | Up, U | 1.10 | |
| 3 | Down, D | 0.97 | |
| 4 | Interest rate, R | 1.06 | |
| 5 | | | |
| 6 | **State prices** | | |
| 7 | $q_U$ | 0.6531 | <-- =(B4-B3)/(B4*(B2-B3)) |
| 8 | $q_D$ | 0.2903 | <-- =(B2-B4)/(B4*(B2-B3)) |
| 9 | | | |
| 10 | **Check: Confirm that state prices actually price the stock and the bond** | | |
| 11 | Pricing the stock: $1 = q_U*U+q_D*D$? | 1 | <-- =B7*B2+B8*B3 |
| 12 | Pricing the bond: $1/R = q_U+q_D$ ? | 1.06 | <-- =1/(B7+B8) |

# Using state prices to price the call and put

| | A | B | C |
|---|---|---|---|
| 1 | **BINOMIAL OPTION PRICING WITH STATE PRICES IN A ONE-PERIOD (TWO-DATE) MODEL** | | |
| 2 | Up, U | 1.10 | |
| 3 | Down, D | 0.97 | |
| 4 | Interest rate, R | 1.06 | |
| 5 | Initial stock price, S | 50.00 | |
| 6 | Option exercise price, X | 50.00 | |
| 7 | | | |
| 8 | **State prices** | | |
| 9 | $q_U$ | 0.6531 | <--  =(B4-B3)/(B4*(B2-B3)) |
| 10 | $q_D$ | 0.2903 | <--  =(B2-B4)/(B4*(B2-B3)) |
| 11 | | | |
| 12 | **Pricing the call and the put** | | |
| 13 | Call price | 3.2656 | <--  =B9*MAX(B5*B2-B6,0)+B10*MAX(B5*B3-B6,0) |
| 14 | Put price | 0.4354 | <--  =B9*MAX(B6-B5*B2,0)+B10*MAX(B6-B5*B3,0) |
| 15 | | | |
| 16 | **Put-call parity** | | |
| 17 | Stock + put | 50.4354 | <--  =B5+B14 |
| 18 | Call + PV(X) | 50.4354 | <--  =B13+B6/B4 |
| 19 | | | |
| 20 | **Note about PV(X) in put-call parity:** In the continuous-time framework (the standard Black-Scholes framework), PV(X) = X*Exp(-r*T).  Because the framework here is discrete time, PV(X) is also discrete-time:  PV(X)=X/(1+r)=X/R. | | |

**Using the state prices gives the same option prices (rows 13/14)**

Using risk-neutral probabilities

# MULTIPLE PERIODS OPTION PRICING

- We seek to evaluate a call at 6 months (3 periods of 2 months) with the exercise price being X = 95.
- The underlying asset is listed currently at $100.
- The risk-free interest rate is r = 2%.
- The factors U and D are set at 1.25 and 0.8.

- The value of the option is the expected value of the different states (Up and Down) weighted for one unit of risk (measured by the risk-free rate) :

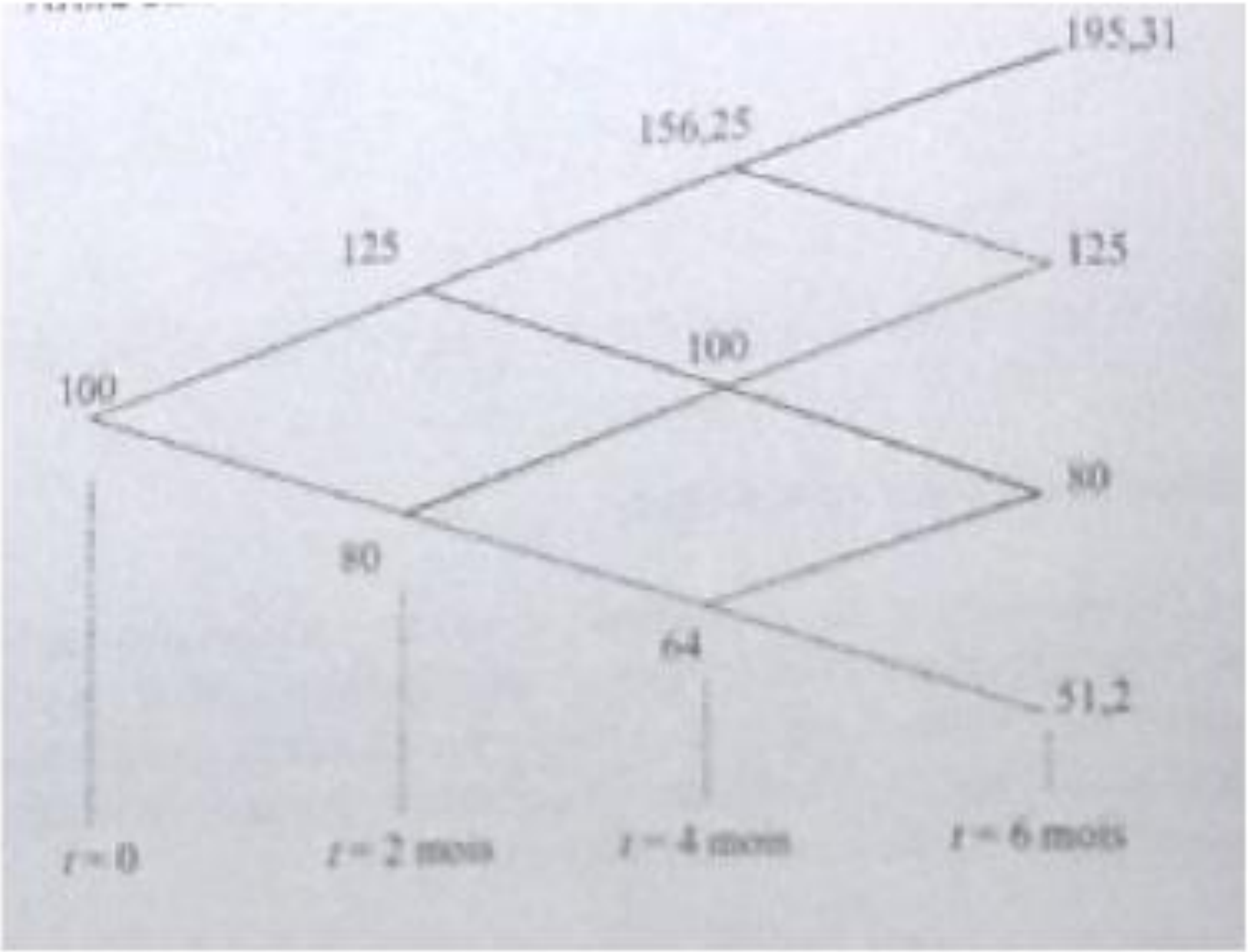
$$g = R^{-\Delta t}\left(q g_u + (1-q) g_d\right)$$

- The risk-neutral probability can be represented by the following formula :

$$q = \frac{R^{\Delta t} - d}{u - d}$$

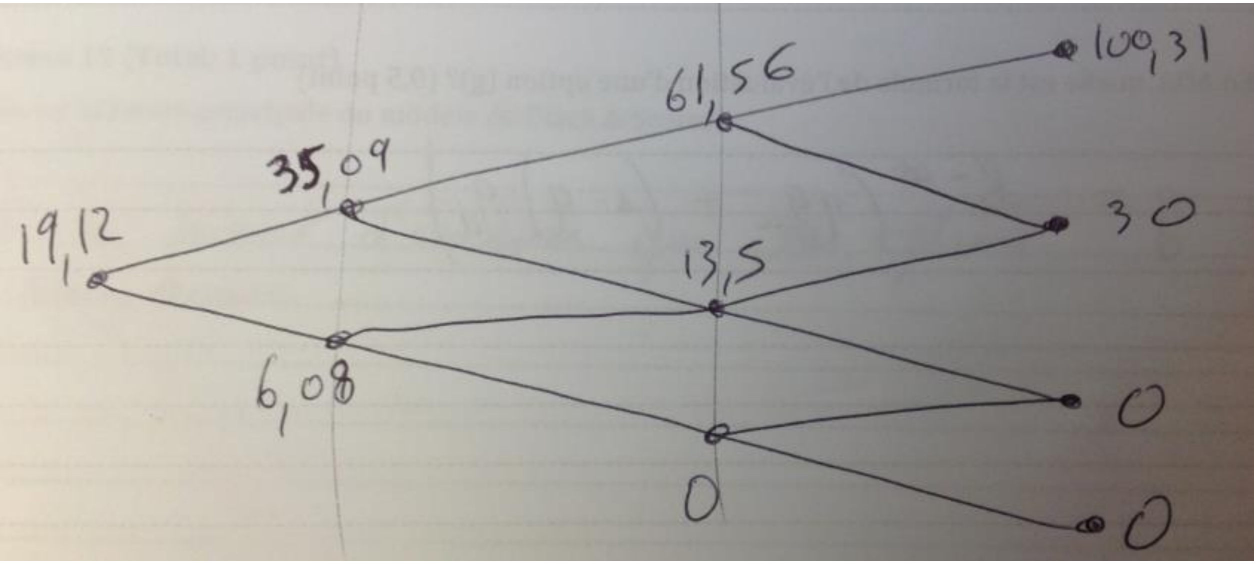
$$\begin{cases} q = \dfrac{1,02^{1/6} - 0,8}{1,25 - 0,8} = 0,4518 \\ (1 - q) = 0,5482 \end{cases}$$

with $\Delta t = T / n$ (T is the time of option exercise in years, and n is the number of subdivisions of T. For instance, if the time of option exercise is 6 months, then T=1/2, and if the 6 months are divided into 3 periods of 2 months, then n=3. Hence, $\Delta t = 1/6$)


195,31
156,25
125
125
100
100
80
80
64
51,2
t = 0
t = 2 mois
t = 4 mois
t = 6 mois

- With an exercise price of X=95 and:

$$g = R^{-\Delta t}\left(qg_u + (1-q)g_d\right)$$




- 61,56=1,02^-(1/6)*(0,4518*100,31+(1-0,4518)*30)

# FROM BINOMIAL OPTION PRICING TO BLACK AND SCHOLES

# Convergence of binomial to Black-Scholes formula

- Use an approximation to the lognormal

$$\Delta t = T / n \qquad R = e^{r\Delta t}$$

$$U = 1 + up = e^{\sigma\sqrt{\Delta t}} \qquad D = 1 + down = e^{-\sigma\sqrt{\Delta t}}$$

- As $\Delta t \to 0$, this converges to the lognormal

| | A | B | C |
|---|---|---|---|
| 1 | **BLACK-SCHOLES AND BINOMIAL PRICING** | | |
| 2 | S | 60 | Current stock price |
| 3 | X | 50 | Option exercise price |
| 4 | T | 0.5000 | Time to option exercise (in years) |
| 5 | r | 8% | Annual interest rate |
| 6 | Sigma | 30% | Riskiness of stock |
| 7 | n | 20 | Number of subdivisions of T |
| 8 | | | |
| 9 | Δt = T/n | 0.0250 | <-- =B4/B7 |
| 10 | Up, U | 1.0486 | <-- =EXP(B6*SQRT(B9)) |
| 11 | Down, D | 0.9537 | <-- =EXP(-B6*SQRT(B9)) |
| 12 | Interest rate, R | 1.0020 | <-- =EXP(B5*B9) |
| 13 | | | |
| 14 | Binomial European call | 12.8055 | <-- =binomial_eur_call(B10,B11,B12,B2,B3,B7) |
| 15 | Black-Scholes call | 12.8226 | <-- =BSCall(B2,B3,B4,B5,B6) |

**In this spreadsheet, we use binomial_eur_call and compare it to the BSCall function from Chapter 19 (OK, so we're a little ahead of ourselves!). As you can see, they are very close.**

| | A | B | C | D |
|---|---|---|---|---|
| 1 | **BLACK-SCHOLES AND BINOMIAL PRICING: CONVERGENCE** | | | |
| 2 | S | 60 | Current stock price | |
| 3 | X | 50 | Option exercise price | |
| 4 | T | 0.5000 | Time to option exercise (in years) | |
| 5 | r | 8% | Annual interest rate | |
| 6 | Sigma | 30% | Riskiness of stock | |
| 7 | n | 20 | Number of subdivisions of T | |
| 8 | | | | |
| 9 | t = T/n | 0.0250 | <-- =B4/B7 | |
| 10 | Up, U | 1.0486 | <-- =EXP(B6*SQRT(B9)) | |
| 11 | Down, D | 0.9537 | <-- =EXP(-B6*SQRT(B9)) | |
| 12 | Interest rate, R | 1.0020 | <-- =EXP(B5*B9) | |
| 13 | | | | |
| 14 | Binomial European call | 12.8055 | <-- =binomial_eur_call(B10,B11,B12,B2,B3,B7) | |
| 15 | Black-Scholes call | 12.8226 | <-- =BSCall(B2,B3,B4,B5,B6) | |
| 16 | | | | |
| 17 | **Data table: Binomial price vs Black-Scholes** | | | |
| 18 | **n, number of subdivisions of T** | **Binomial price** | **Black-Scholes price** | |
| 19 | | 12.8055 | 12.8226 | <-- Data table headers |
| 20 | 10 | 12.8593 | 12.8226 | |
| 21 | 50 | 12.8108 | 12.8226 | |
| 22 | 75 | 12.8238 | 12.8226 | |
| 23 | 100 | 12.8255 | 12.8226 | |
| 24 | 125 | 12.8251 | 12.8226 | |
| 25 | 150 | 12.8240 | 12.8226 | |
| 26 | 175 | 12.8226 | 12.8226 | |
| 27 | 200 | 12.8205 | 12.8226 | |
| 28 | 225 | 12.8204 | 12.8226 | |
| 29 | 250 | 12.8230 | 12.8226 | |
| 30 | 275 | 12.8243 | 12.8226 | |
| 31 | 300 | 12.8243 | 12.8226 | |
| 32 | 325 | 12.8232 | 12.8226 | |
| 33 | 350 | 12.8210 | 12.8226 | |
| 34 | 375 | 12.8226 | 12.8226 | |
| 35 | 400 | 12.8238 | 12.8226 | |
| 36 | 425 | 12.8236 | 12.8226 | |
| 37 | 450 | 12.8221 | 12.8226 | |
| 38 | 475 | 12.8223 | 12.8226 | |
| 39 | 500 | 12.8236 | 12.8226 | |

**Now we let $\Delta T \rightarrow 0$, and you can see the convergence.**