Chapter 9: Efficient portfolio theorems

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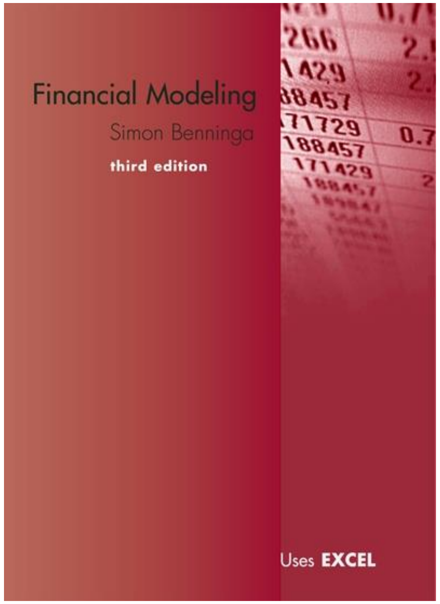


Notation (1)

N risky assets.

Vector of expected returns:

$$E(r) = \begin{bmatrix} E(r_1) \\ E(r_2) \\ \vdots \\ E(r_N) \end{bmatrix}$$

Variance-covariance matrix:

$$S = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1N} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2N} \\ \vdots & & & \\ \sigma_{N1} & \sigma_{N2} & \cdots & \sigma_{NN} \end{bmatrix}$$

Notation (2)

A *portfolio* of risky assets is a set of proportions x_i which sum to 1.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}, \quad \sum_{i=1}^{N} x_i = 1$$

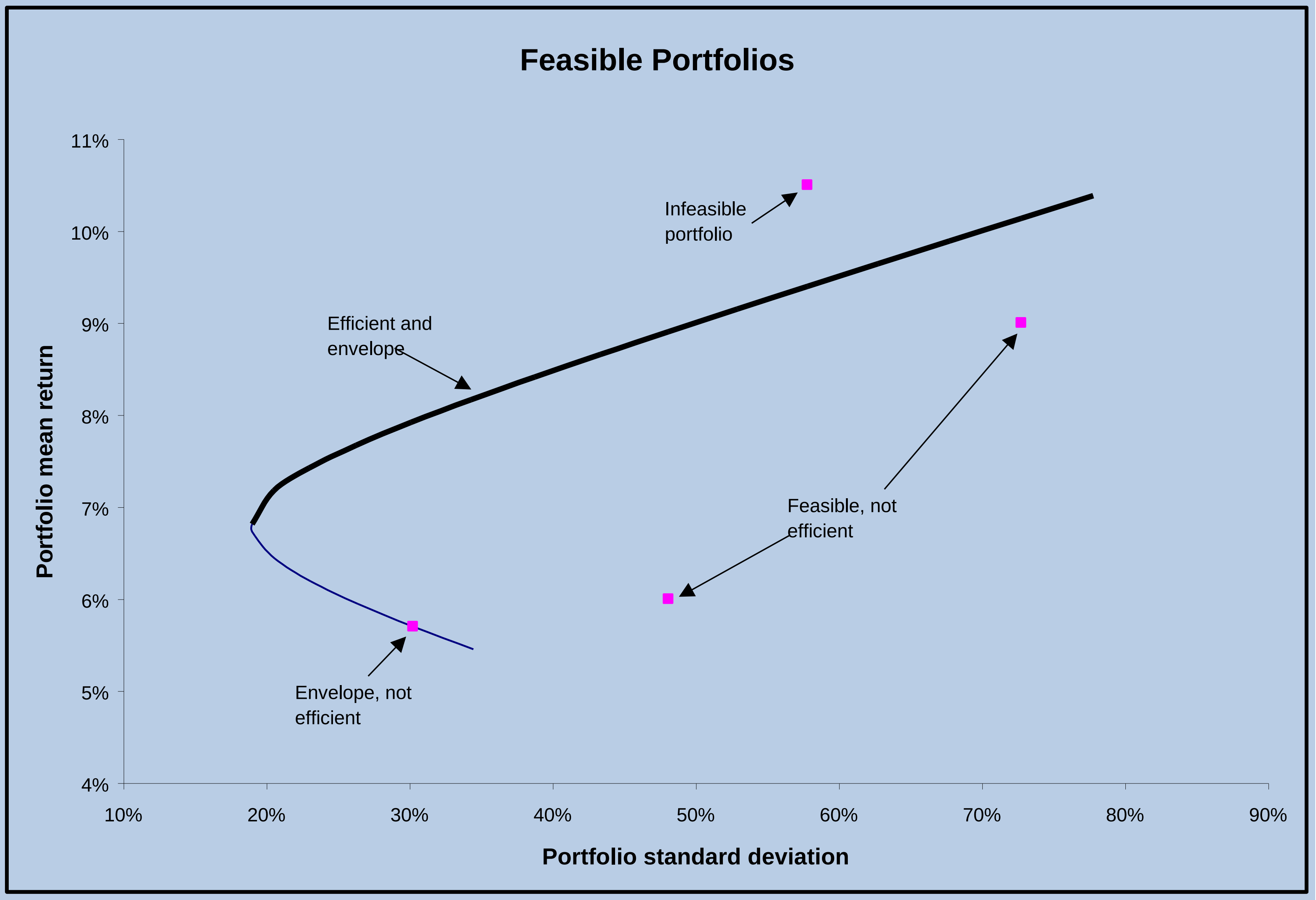
The portfolio expected return is:

$$E\left(r_x\right) = x^T \cdot R \equiv \sum_{i=1}^{N} x_i E\left(r_i\right)$$

Notation (3)

The portfolio variance is

$$\sigma_x^2 = x^T S x = \sum_{i=1}^{N} \sum_{j=1}^{N} x_i x_j \sigma_{ij}$$


Feasible Portfolios
Portfolio mean return
11%
10%
9%
8%
7%
6%
5%
4%
10%
20%
30%
40%
50%
60%
70%
80%
90%
Portfolio standard deviation
Infeasible portfolio
Efficient and envelope
Feasible, not efficient
Envelope, not efficient

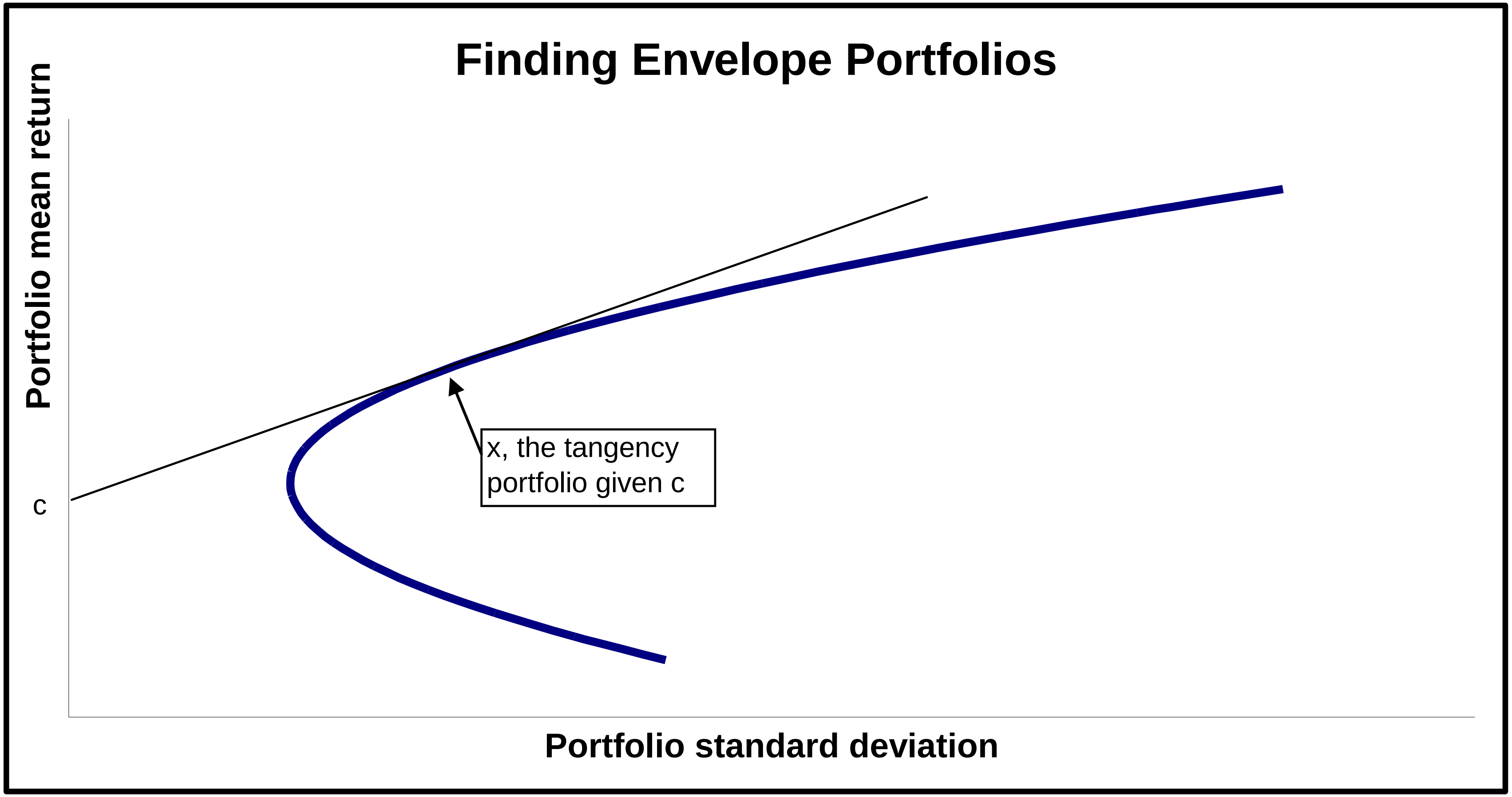
FIVE PROPOSITIONS ABOUT THE ENVELOPE PORTFOLIOS

Five propositions about envelope portfolios

Proposition 1 (Merton 1973): All envelope portfolios solve the following equation:

$$x = \frac{S^{-1}\left\{E\left(r\right)-c\right\}}{\sum S^{-1}\left\{E\left(r\right)-c\right\}}$$

Here *c* is an arbitrary constant.



Proposition 1 gives a method of finding the tangency portfolio.

Computing two efficient portfolios

	A	B	C	D	E	F	G	H
1	CALCULATING THE EFFICIENT FRONTIER							
2	Variance-covariance matrix					Expected returns E(r)	Expected minus constant E(r) - c	
3	0.40	0.03	0.02	0.00		0.06	0.02	<-- =F3-$B$8
4	0.03	0.20	0.00	-0.06		0.05	0.01	<-- =F4-$B$8
5	0.02	0.00	0.30	0.03		0.07	0.03	<-- =F5-$B$8
6	0.00	-0.06	0.03	0.10		0.08	0.04	<-- =F6-$B$8
7								
8	Constant	0.04						
9								
10	Computing an envelope portfolio with constant = 0							
11	z					Envelope portfolio x		
12	0.1019	<-- {=MMULT(MINVERSE(A3:D6),F3:F6)}				0.0540	<-- =A12/SUM($A$12:$A$15)	
13	0.5657					0.2998		
14	0.1141					0.0605		
15	1.1052					0.5857		
16					Sum	1.0000	<-- =SUM(F12:F15)	
17								
18	Computing an envelope portfolio with constant = 0.04							
19	z					Envelope portfolio y		
20	0.0330	<-- {=MMULT(MINVERSE(A3:D6),G3:G6)}				0.0423	<-- =A20/SUM($A$20:$A$23)	
21	0.1959					0.2514		
22	0.0468					0.0601		
23	0.5035					0.6462		
24					Sum	1.0000	<-- =SUM(F20:F23)	

Here we compute the efficient portfolio in 2 steps.

In step 1 we compute a portfolio *z*:

$$z = S^{-1}\left\{E\left(r\right) - c\right\}$$

In step 2, we normalize *z* to sum to 1, computing two efficient portfolios, *x* and *y*.

Proposition 2 (Merton): The convex combination of any two envelope portfolios is also an envelope portfolio.

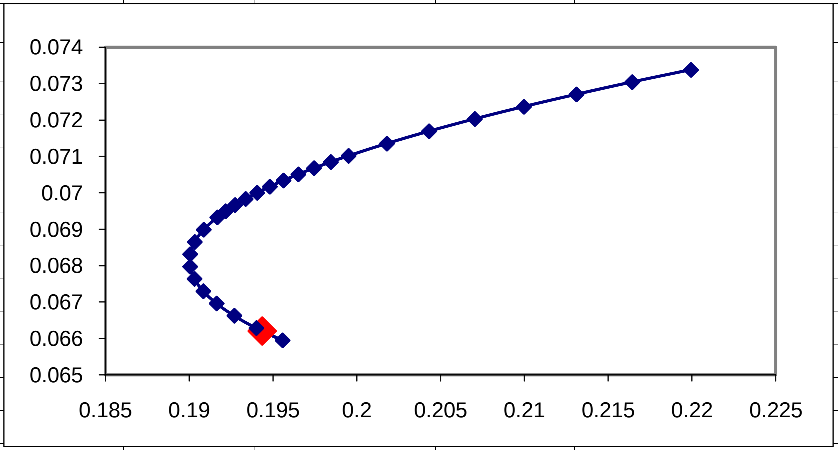
Thus: if *x* and *y* are envelope portfolios, so is

$$\lambda x + (1-\lambda) y = \begin{Bmatrix} \lambda x_1 + (1-\lambda) y_1 \\ \cdots \\ \lambda x_N + (1-\lambda) y_N \end{Bmatrix}$$

SOME c's CAN LEAD TO INEFFICIENT PORTFOLIOS

The portfolio *x* determined by the constant c=0.11 is inefficient

	A	B	C	D	E	F	G
2	**Variance-covariance matrix**					**Expected returns E(r)**	
3	0.40	0.03	0.02	0.00		0.06	
4	0.03	0.20	0.00	-0.06		0.05	
5	0.02	0.00	0.30	0.03		0.07	
6	0.00	-0.06	0.03	0.10		0.08	
7							
8	Constant	0.11					
9							
10	**Computing an envelope portfolio with constant = 0.11**						
11	**z**					**Envelope portfolio x**	
12	-0.0876	<-- {=MMULT(MINVERSE(A3:D6),F3:F6-B8)}				0.0755	<-- {=A12:A15/SUM(A12:A15)}
13	-0.4514					0.3893	
14	-0.0710					0.0613	
15	-0.5495					0.4739	
16					Sum	1.0000	<-- =SUM(F12:F15)
17							
18							
19	$E(r_x)$	0.0662					
20	σ_x	0.1944					



Proposition 1 defines a procedure for finding an envelope portfolio.

This portfolio can be efficient or inefficient.

Computing the envelope (1)

- Propositions 1 and 2 show how to compute the envelope (the efficient frontier)
- First—compute two envelope portfolios *x* and *y*. Do this with two arbitrary constants c_1 and c_2 .
- Second—compute the <u>mean</u> and <u>standard deviation</u> of all convex combinations of *x* and *y* .

Computing the envelope (2)

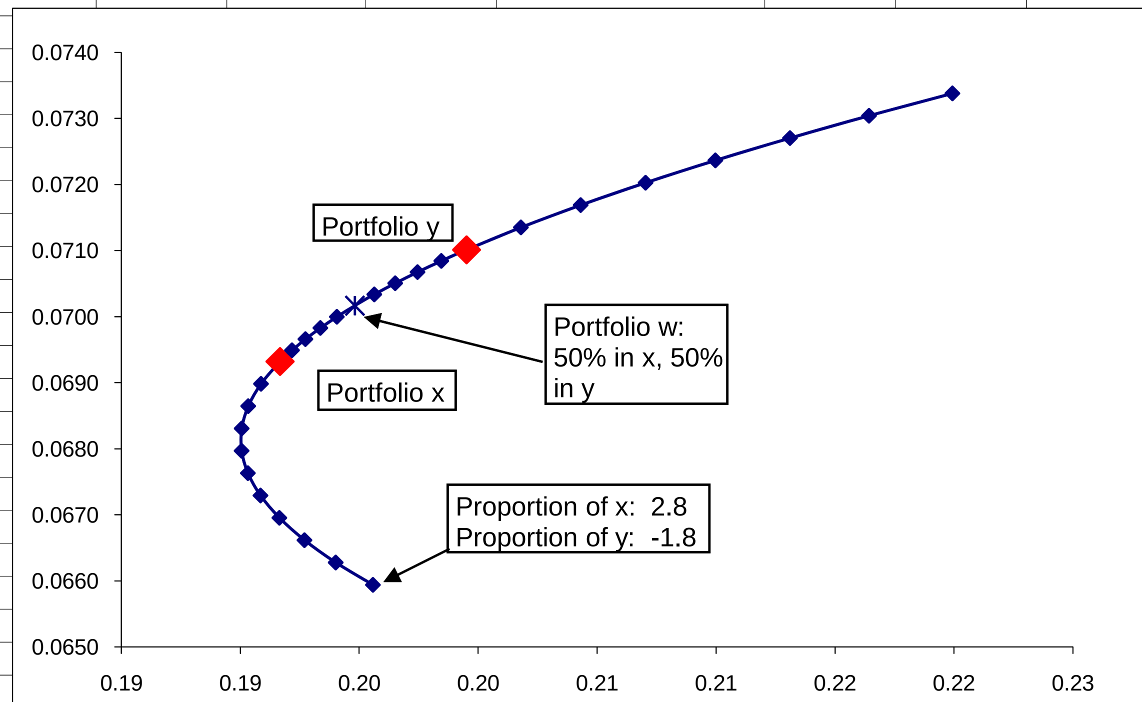
	A	B	C	D	E	F	G	H	I	J	K
26	**E(x)**	0.0693			**E(x)**	0.0710	<-- {=MMULT(TRANSPOSE(F20:F23),F3:F6)}				
27	**Var(x)**	0.0367			**Var(y)**	0.0398	<-- {=MMULT(MMULT(TRANSPOSE(F20:F23),A3:D6),F20:F23)}				
28	**Sigma(x)**	0.1917			**Sigma(y)**	0.1995	<-- =SQRT(F27)				
29											
30		**Cov(x,y)**	0.0376	<-- {=MMULT(MMULT(TRANSPOSE(F12:F15),A3:D6),F20:F23)}							
31		**Corr(x,y)**	0.9842	<-- =C30/(B28*F28)							
32											
33											
34	**A single portfolio calculation**										
35	Proportion of x	0.3									
36	$E(r_p)$	7.05%	<-- =B35*B26+(1-B35)*F26								
37	σ_p	19.65%	<-- =SQRT(B35^2*B27+(1-B35)^2*F27+2*B35*(1-B35)*C30)								

- Portfolios *x* and *y* are envelope portfolios.
- Rows 26:31 compute the relevant statistics
- Rows 35:37 compute the expected return and standard deviation of a single portfolio in which proportion of x = 0.3 and the proportion of y = 0.7.

In the next slide we repeat the computations in a **Data Table** to compute the whole envelope.

Computing the envelope (3)

	A	B	C	D
40	**Data table: we vary the proportion of x to produce a graph of the frontier**			
41	Proportion of x	**Sigma**	**Return**	
42		0.1965	0.0705	<-- Data table header refers to cells B36 and B35
43	-1.400	0.2199	0.0734	
44	-1.200	0.2164	0.0730	
45	-1.000	0.2131	0.0727	
46	-0.800	0.2100	0.0724	
47	-0.600	0.2070	0.0720	
48	-0.400	0.2043	0.0717	
49	-0.200	0.2018	0.0713	
50	0.000	0.1995	0.0710	
51	0.100	0.1984	0.0708	
52	0.200	0.1974	0.0707	
53	0.300	0.1965	0.0705	
54	0.400	0.1956	0.0703	
55	0.500	0.1948	0.0702	
56	0.600	0.1941	0.0700	
57	0.700	0.1934	0.0698	
58	0.800	0.1927	0.0697	
59	0.900	0.1922	0.0695	
60	1.000	0.1917	0.0693	
61	1.200	0.1909	0.0690	
62	1.400	0.1903	0.0686	
63	1.600	0.1901	0.0683	
64	1.800	0.1901	0.0680	
65	2.000	0.1903	0.0676	
66	2.200	0.1908	0.0673	
67	2.400	0.1916	0.0670	



Any two constants lead to same efficient frontier

- An efficient frontier is a combination of two efficient portfolios.
- Doesn't matter which c's you use to compute the two efficient portfolios—you get the same efficient frontier .

	A	B	C	D	E	F	G	H
1	**FINDING EFFICIENT PORTFOLIOS IN ONE STEP**							
2	**Variance-covariance matrix**					**Expected returns E(r)**		
3	0.40	0.03	0.02	0.00		0.06		
4	0.03	0.20	0.00	-0.06		0.05		
5	0.02	0.00	0.30	0.03		0.07		
6	0.00	-0.06	0.03	0.10		0.08		
7								
8	Constant	0.05						
9								
10	**Envelope portfolio**							
11	0.0314	<-- {=MMULT(MINVERSE(A3:D6),F3:F6-B8)/SUM(MMULT(MINVERSE(A3:D6),F3:F6-B8))}						
12	0.2059							
13	0.0597							
14	0.7031							
15								
16	Portfolio expected return, $E(r_x)$	7.26%	<-- =SUMPRODUCT(A11:A14,F3:F6)					
17	Portfolio standard devation, σ_x	21.21%	<-- {=SQRT(MMULT(MMULT(TRANSPOSE(A11:A14),A3:D6),A11:A14))}					

Creative use of array functions: Use to find efficient portfolios in one step (cells A11:A14).

Proposition 3 (Black, 1972): If portfolio *y* is envelope, then all other portfolios are related to *y* through the following linear relationship:

$$E(r_x) = c + \beta_x \left[E(r_y) - c \right]$$

where

$$\beta_x = \frac{Cov(x, y)}{\sigma_y^2}$$

Proposition 4 (Black): If the linear relationship below holds, then portfolio *y* is an envelope portfolio.

$$E\left(r_x\right) = c + \beta_x\left[E\left(r_y\right) - c\right]$$

where

$$\beta_x = \frac{Cov(x, y)}{\sigma_y^2}$$

Proposition 5: If there exists a risk-free asset with return r_f, then the standard security market line (SML) relationship holds:

$$E\left(r_x\right)=r_f+\beta_x\left[E\left(r_x\right)-r_f\right]$$

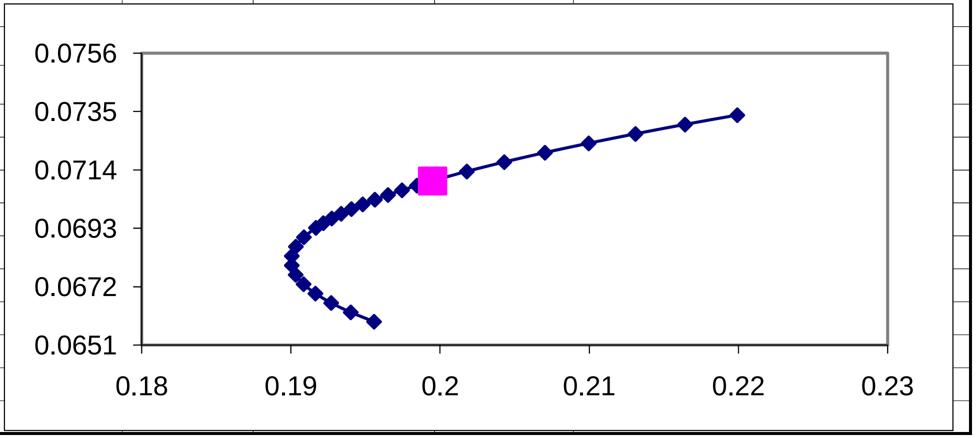
$$\textit{where}$$

$$\beta_x=\frac{\operatorname{cov}(x,M)}{\sigma_M^2}$$

IF $c = r_f$ AND THE OPTIMIZING PORTFOLIO IS EFFICIENT, THEN THE ENVELOPE PORTFOLIO IS OPTIMAL

The portfolio *x* determined by the constant c= 4% is optimal

	A	B	C	D	E	F	G
2	**Variance-covariance matrix**					**Expected returns**	
3	0.40	0.03	0.02	0.00		0.06	
4	0.03	0.20	0.00	-0.06		0.05	
5	0.02	0.00	0.30	0.03		0.07	
6	0.00	-0.06	0.03	0.10		0.08	
7							
8	Constant	0.04					
9							
10	**Computing an envelope portfolio with constant = 0.04**						
11	**z**					**Envelope portfolio x**	
12	0.0330	<-- {=MMULT(MINVERSE(A3:D6),F3:F6-B8)}				0.0423	<-- {=A12:A15/SUM(A12:A15)}
13	0.1959					0.2514	
14	0.0468					0.0601	
15	0.5035					0.6462	
16					Sum	1.0000	<-- =SUM(F12:F15)
17							
18							
19	$E(r_x)$	0.0710					
20	σ_x	0.1995					



	A	B	C	D	E	F	G
1	**ILLUSTRATING PROPOSITIONS 3-5**						
2	**Dates**	**Asset 1**	**Asset 2**	**Asset 3**	**Asset 4**		**Efficient portfolio w**
3	1	-6.63%	-2.49%	-4.27%	11.72%		-1.82%
4	2	8.53%	2.44%	-3.15%	-8.33%		0.99%
5	3	1.79%	4.46%	1.92%	19.18%		5.47%
6	4	7.25%	17.90%	-6.53%	-7.41%		3.65%
7	5	0.75%	-8.22%	-1.76%	-1.44%		-2.51%
8	6	-1.57%	0.83%	12.88%	-5.92%		2.16%
9	7	-2.10%	5.14%	13.41%	-0.46%		4.14%
10							
11	Mean	1.15%	2.87%	1.79%	1.05%	<-- =AVERAGE(E3:E9)	1.73%
12							
13	**Variance-covariance matrix**						
14		Asset 1	Asset 2	Asset 3	Asset 4		
15	Asset 1	0.0024	0.0019	-0.0015	-0.0024		
16	Asset 2	0.0019	0.0056	-0.0007	-0.0016	Cells B15:E18 contain the formula	
17	Asset 3	-0.0015	-0.0007	0.0057	-0.0005	{=MMULT(TRANSPOSE(B3:E9-B11:E11),B3:E9-	
18	Asset 4	-0.0024	-0.0016	-0.0005	0.0094	B11:E11)/7}	
19							
20	**Finding an efficient portfolio w**						
21	Constant	0.50%					
22							
23	Asset 1	0.3129					
24	Asset 2	0.2464	<--	Cells B23:B26 contain the formula			
25	Asset 3	0.2690		{=MMULT(MINVERSE(B15:E18),TRANSPOSE(B11:E11)-			
26	Asset 4	0.1717		B21)/SUM(MMULT(MINVERSE(B15:E18),TRANSPOSE(B11:E11)-B21))}			

The efficient portfolio in B23:B26 depends on the constant in cell B21.
The returns in G3:G9 are the annual returns of this efficient portfolio.

For example, cell G3:
-1.82%=0.3129*-6.63%+0.2464*-2.49%+0.2690*-4.27%+0.1717*11.72%

	A	B	C	D	E	F	G
29	**Implementing propositions 3-5--finding the SML**						
30	Step 1: Regress each asset's returns on those of the efficient portfolio w						
31		**Asset 1**	**Asset 2**	**Asset 3**	**Asset 4**		
32	Alpha	0.0024	-0.0047	-0.0002	0.0028	<-- =INTERCEPT(E3:E9,$G$3:$G$9)	
33	Beta	0.5284	1.9301	1.0490	0.4478	<-- =SLOPE(E3:E9,$G$3:$G$9)	
34	R-squared	0.0897	0.5241	0.1505	0.0167	<-- =RSQ(E3:E9,$G$3:$G$9)	
35							
36	Step 2: Regress the asset mean returns on their betas						
37	Intercept	0.005	<-- =INTERCEPT(B11:E11,B33:E33)				
38	Slope	0.0123	<-- =SLOPE(B11:E11,B33:E33)				
39	R-squared	1.0000	<-- =RSQ(B11:E11,B33:E33)				
40							
41	**Check Propositions 3 & 4: Step 2 coefficients should be:** **Intercept = c, Slope = $E(r_w)$ - c**						
42	Intercept = c ?	yes	<-- =IF(B36=B20,"yes","no")				
43	Slope = $E(r_w)$ - c ?	yes	<-- =IF(B38=G11-B21,"yes","no")				

Rows 30-34: Regressing each asset's returns on the efficient portfolio.

Rows 36-39: Regressing asset mean returns on their betas. **The regression is perfect**. This is what Propositions 3-5 guarantee—because the regression is done with respect to an efficient portfolio.

CONCLUSION

Some philosophy (I)

- CAPM is the search for a linear return relation between expected return and portfolio risk .
- This is the meaning of the classic SML:
- Propositions 3,4 say: There is a linear relation if and only if portfolios are regressed on an efficient portfolio.

Philosophy (2)

- Classic SML tests are two-step regressions:
  - Pick an index M which might represent the market portfolio.
  - Step 1: Regress each asset's returns on the index M to find the asset's β:

$$r_{it} = \alpha_i + \beta_i r_{Mt}$$

  - Step 2: Regress the asset mean returns on the betas to "find" the SML

The actual market portfolio is the value-weighted portfolio of all risky assets.

$$\bar{r}_i = \gamma_0 + \gamma_1 \beta_i$$

- Question: What's the R^2 of the SML?

Philosophy (3): Roll (1977)

- From Propositions 3,4 we learn that an $R^2 = 100\%$ exists only if the index M is efficient.
- Therefore the real question: Is the true market portfolio efficient?
- Other questions of the same type:
  - Does God exist?
  - ???