An extension to Markowitz

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An extension of Markowitz’s frontier

- Markowitz’s efficient portfolio

Markowitz

- Le's assume three stocks:

A	B	C	D
Observations	Stock 1	Stock 2	Stock 3
1	-0,1	0,3	0,33
2	0,02	0,2	0,6
3	-0,39	0,25	0,48
4	0,9	0,24	0,38
5	0,45	0,19	-0,3

Markowitz

$$E(R_i) = \hat{\mu}_i = \frac{\sum_{j=1}^{n} R_{i,j}}{n}$$

$$\hat{\sigma}_i^2 = \frac{\sum_{j=1}^{n} \left(R_{i,j} - \hat{\mu}_i\right)^2}{n-1} = \begin{cases} Var(B2:B6) \\ Var(C2:C6) \\ Var(D2:D6) \end{cases}$$

	Stock 1	Stock 2	Stock 3
Xi	0,33	0,33	0,33
Expected return	0,176	0,236	0,298
Variance of return	0,25503	0,00193	0,12242

Markowitz

$$E\left(R_p\right) = \sum_{i=1}^{n} X_i R_i = X_1 \hat{A}_1 + X_2 \hat{A}_1 + X_3 \hat{A}_1$$ =MMULT(B14:D14,B15:D15)

$$V\left(R_p\right) = \sigma^2\left(R_p\right) = \sum_{i=1}^{n} X_i^2 V\left(R_i\right) + 2\sum_{i\neq}\sum_{j} X_i X_j \operatorname{cov}\left(R_j, R_i\right)$$

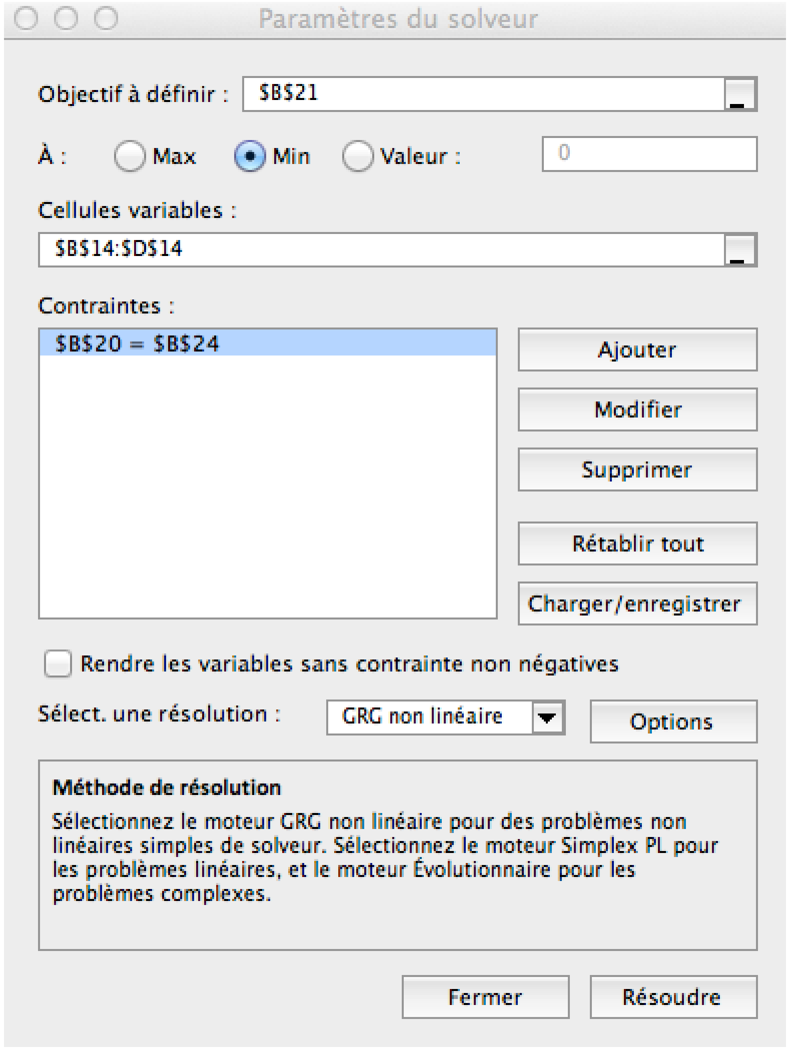
$$= \sum_{i=1}^{n} X_i^2 \sigma_i^2 + 2\sum_{i\neq}\sum_{j} X_i X_j \operatorname{cov}\left(R_j, R_i\right)$$

=MMULT(B14:D14,MMULT(G14:I14,TRANSPOSE(B14:E14)))

Markowitz

Ci,j	Stock 1	Stock 2	Stock 3
Stock 1	0,25503	-0,005936	-0,052688
Stock 2	-0,005936	0,00193	0,004312
Stock 3	-0,052688	0,004312	0,12242

18			
19		Portfolio	
20	E(Rp)	0,236666667	MMULT(B1
21	Var(Rp)	0,030084	MMULT(B1
22			
23			
24	E(Rp)	0,25	
25	Xi:	1,00	
26			

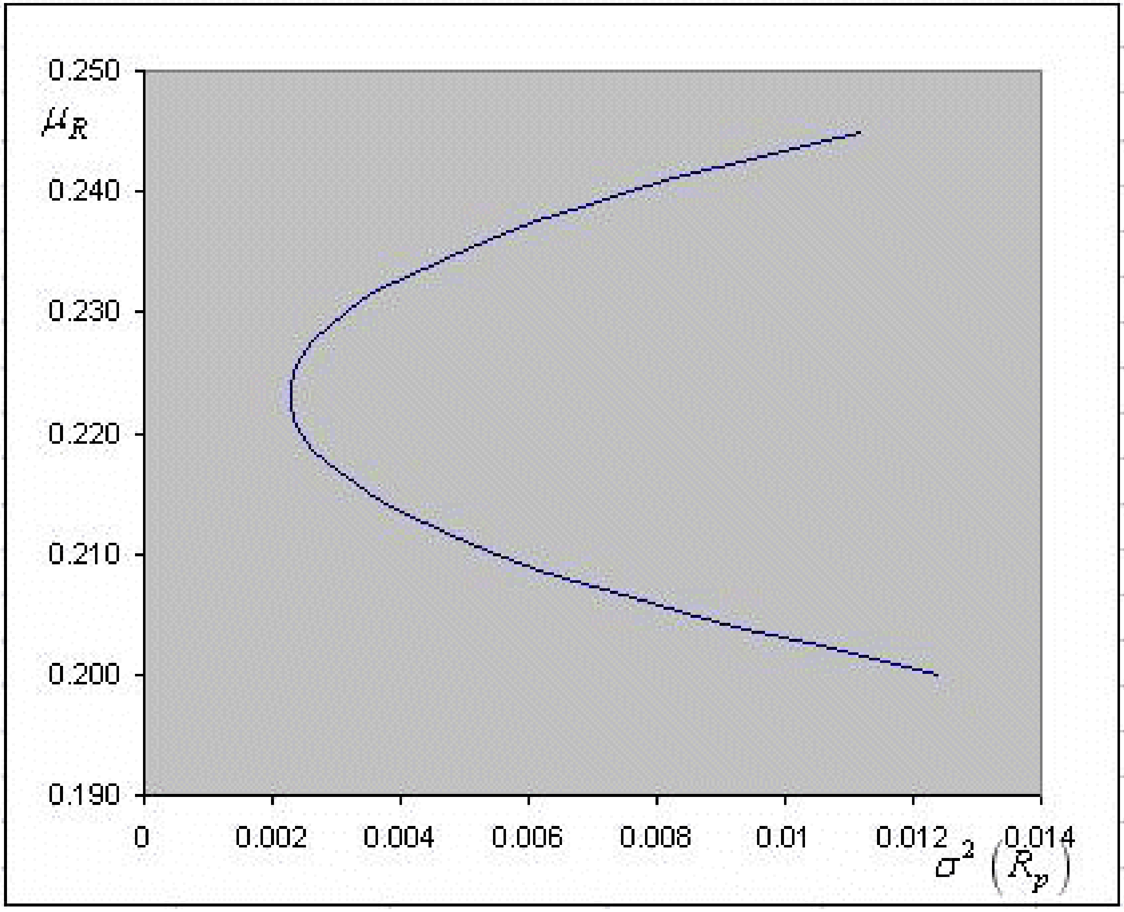


$$\min : \sigma^2\left(R_p\right)$$

$$E\left(R_p\right) = 0.2$$

$$\sum X_i = 1$$

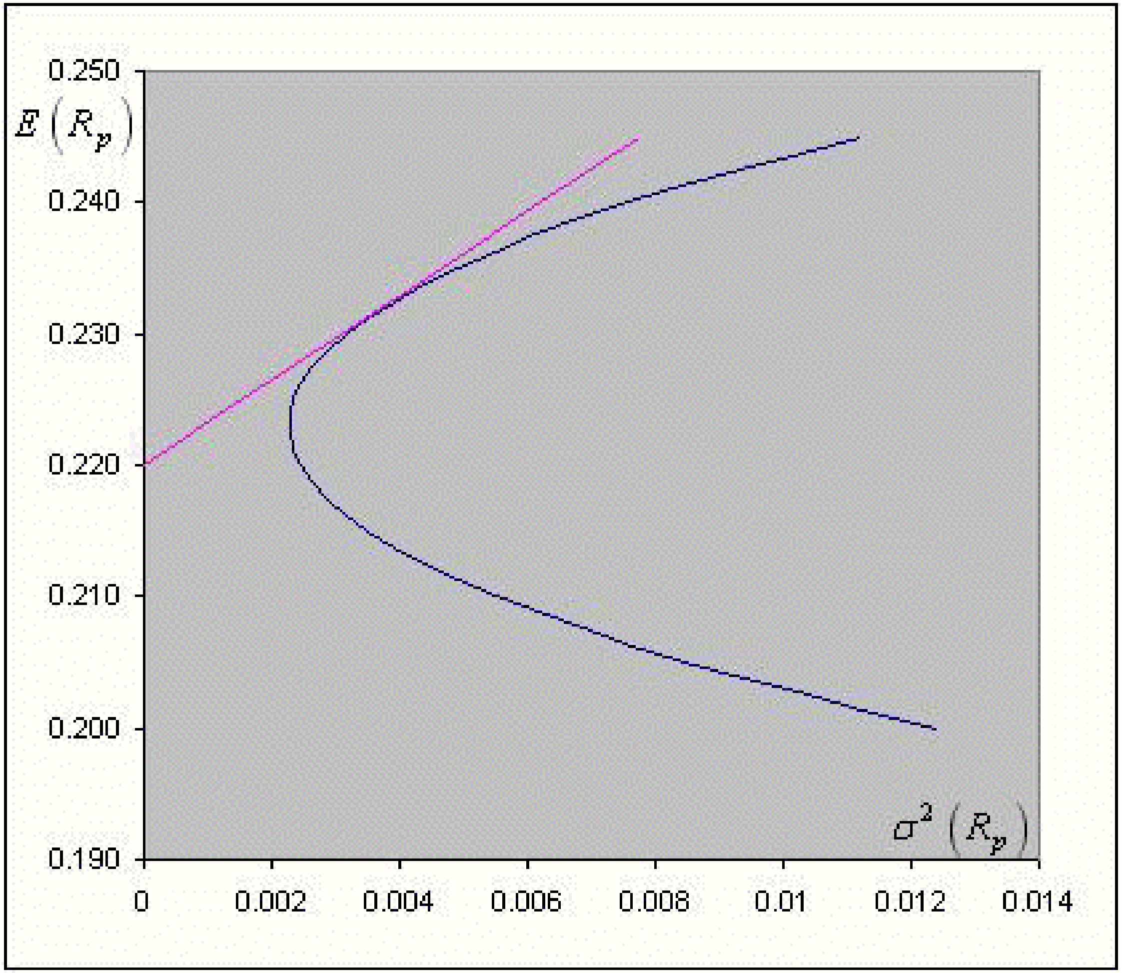
Markowitz


0.250
μ_R
0.240
0.230
0.220
0.210
0.200
0.190
0
0.002
0.004
0.006
0.008
0.01
0.012
0.014
$\sigma^2\left(R_p\right)$

Capital Market Line (CML)

- Risk-free asset

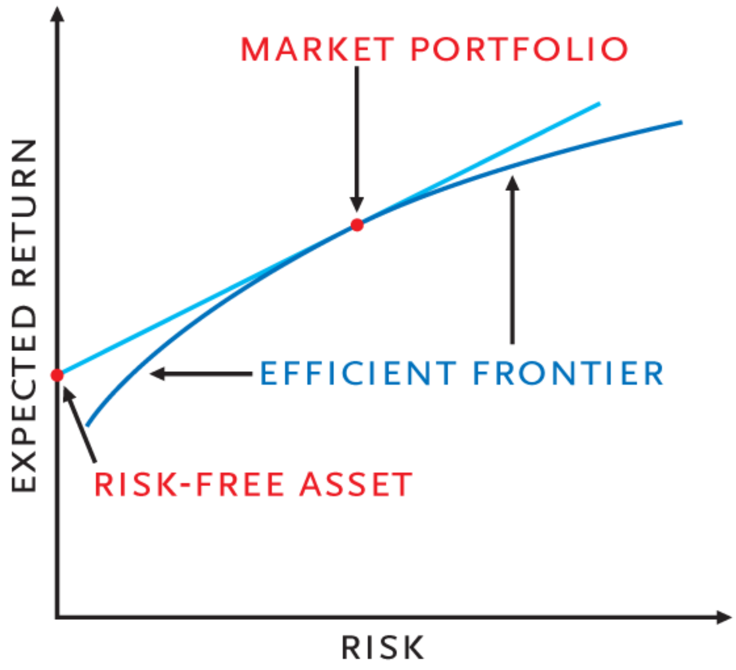
Markowitz

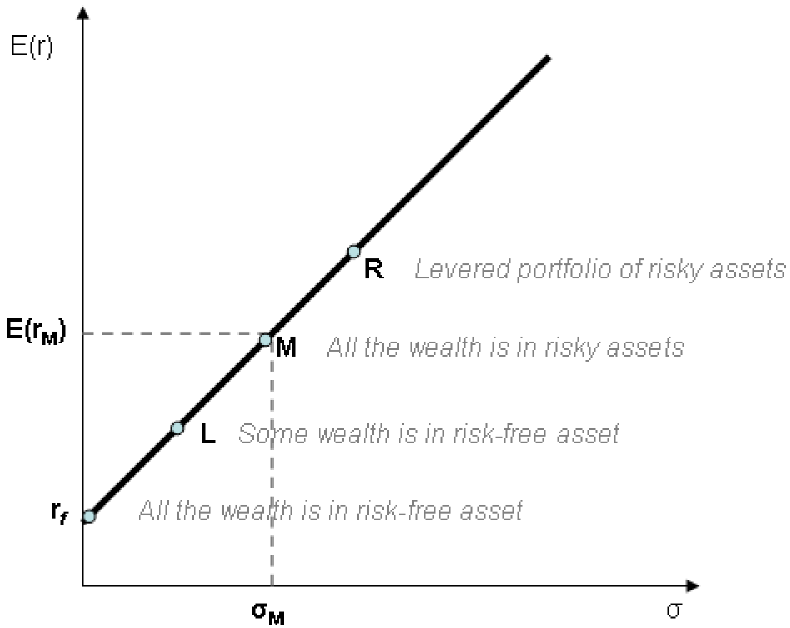


0.250
$E\left(R_p\right)$
0.240
0.230
0.220
0.210
0.200
0.190
$\sigma^2\left(R_p\right)$
0
0.002
0.004
0.006
0.008
0.01
0.012
0.014

Capital Market Line (CML)

$$\text{CML} : E(r) = r_f + \sigma \frac{E(r_M) - r_f}{\sigma_M}.$$





Notation (1)

N risky assets.

Vector of expected returns:

$$E(r)=\begin{bmatrix} E(r_1) \\ E(r_2) \\ \vdots \\ E(r_N) \end{bmatrix}$$

Variance-covariance matrix:

$$S=\begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1N} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2N} \\ \vdots & & & \\ \sigma_{N1} & \sigma_{N2} & \cdots & \sigma_{NN} \end{bmatrix}$$

Notation (2)

A *portfolio* of risky assets is a set of proportions x_i which sum to 1.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}, \quad \sum_{i=1}^{N} x_i = 1$$

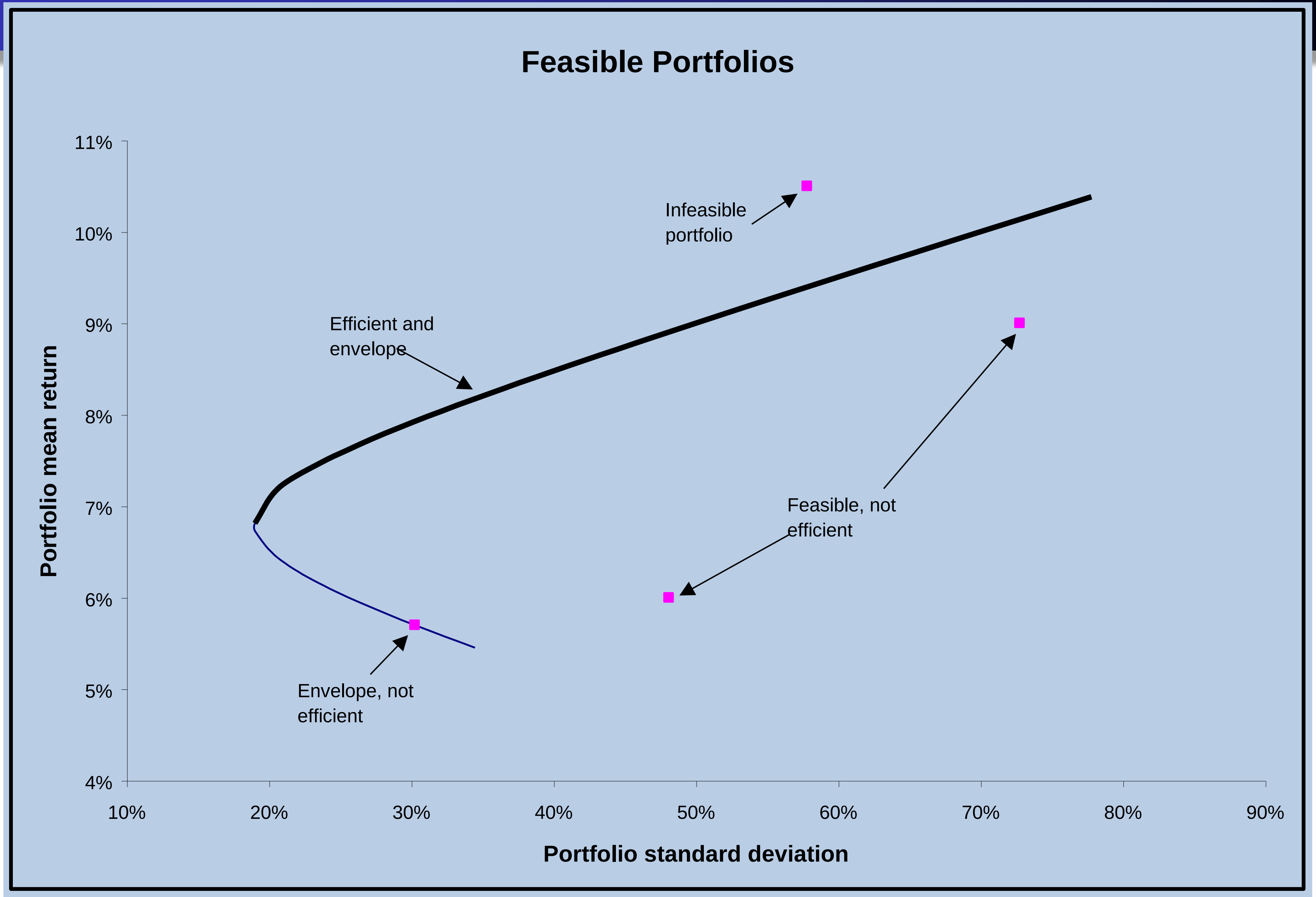
The portfolio expected return is:

$$E\left(r_x\right) = x^T \cdot R \equiv \sum_{i=1}^{N} x_i E\left(r_i\right)$$

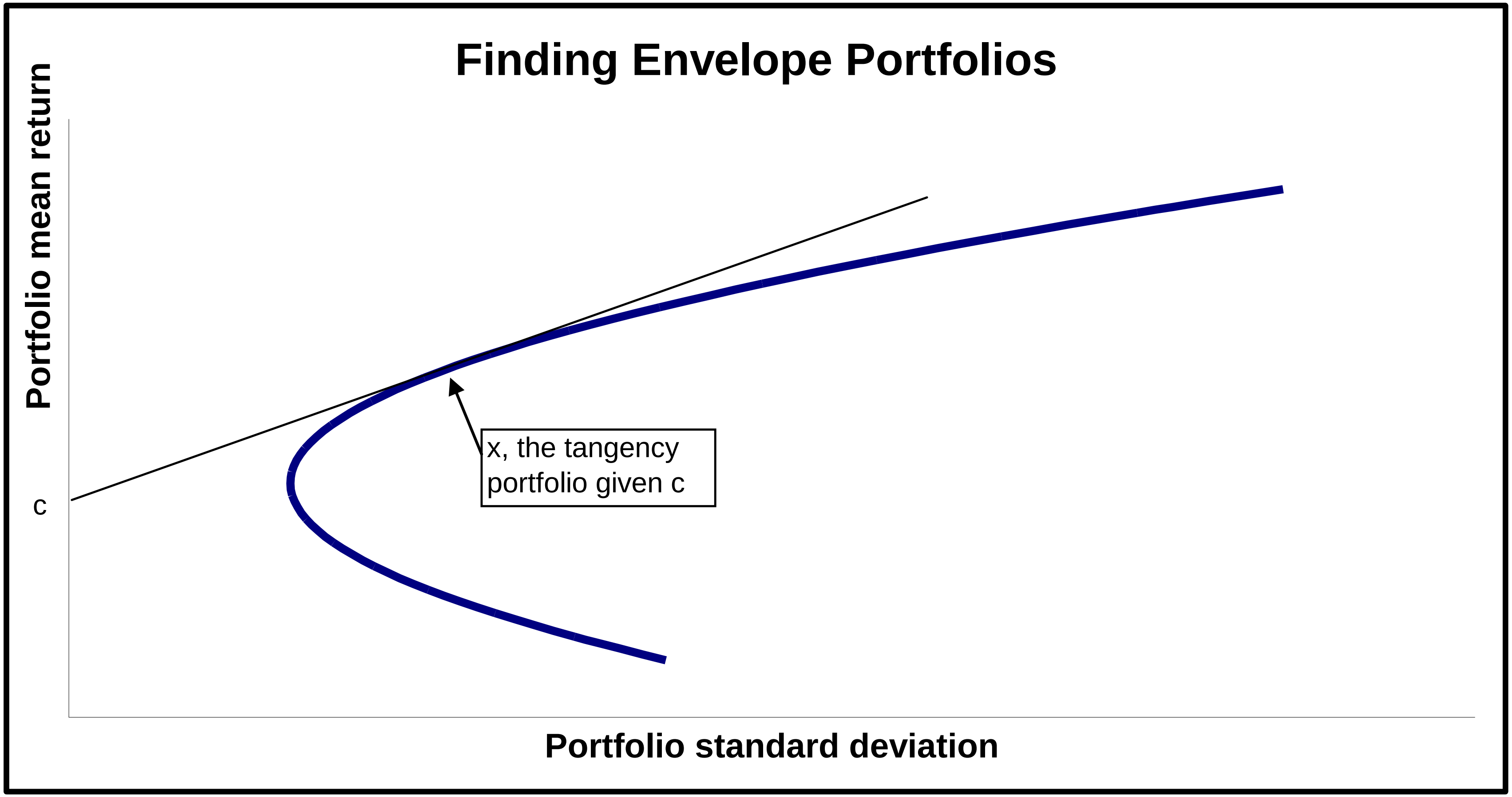
Notation (3)

The portfolio variance is

$$\sigma_x^2 = x^T S x = \sum_{i=1}^{N} \sum_{j=1}^{N} x_i x_j \sigma_{ij}$$

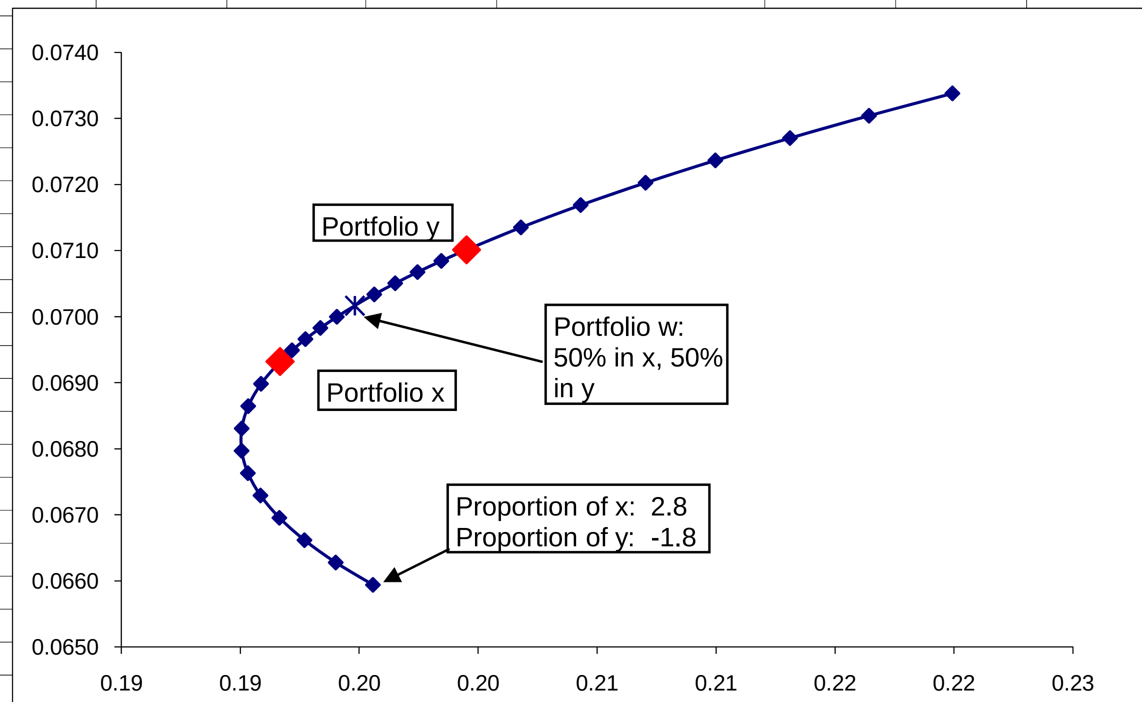


Feasible Portfolios
Portfolio mean return
Portfolio standard deviation
11%
10%
9%
8%
7%
6%
5%
4%
10%
20%
30%
40%
50%
60%
70%
80%
90%
Infeasible portfolio
Efficient and envelope
Feasible, not efficient
Envelope, not efficient


Finding Envelope Portfolios
Portfolio mean return
c
x, the tangency portfolio given c
Portfolio standard deviation

Computing the envelope (3)

	A	B	C	D	E	F	G	H	I	J	K
40	**Data table: we vary the proportion of x to produce a graph of the frontier**										
41	Proportion of x	**Sigma**	**Return**								
42		0.1965	0.0705	<-- Data table header refers to cells B36 and B35							
43	-1.400	0.2199	0.0734								
44	-1.200	0.2164	0.0730								
45	-1.000	0.2131	0.0727								
46	-0.800	0.2100	0.0724								
47	-0.600	0.2070	0.0720								
48	-0.400	0.2043	0.0717								
49	-0.200	0.2018	0.0713								
50	0.000	0.1995	0.0710								
51	0.100	0.1984	0.0708								
52	0.200	0.1974	0.0707								
53	0.300	0.1965	0.0705								
54	0.400	0.1956	0.0703								
55	0.500	0.1948	0.0702								
56	0.600	0.1941	0.0700								
57	0.700	0.1934	0.0698								
58	0.800	0.1927	0.0697								
59	0.900	0.1922	0.0695								
60	1.000	0.1917	0.0693								
61	1.200	0.1909	0.0690								
62	1.400	0.1903	0.0686								
63	1.600	0.1901	0.0683								
64	1.800	0.1901	0.0680								
65	2.000	0.1903	0.0676								
66	2.200	0.1908	0.0673								
67	2.400	0.1916	0.0670								



Proposition (Black): If the linear relationship below holds, then portfolio *y* is an envelope portfolio.

$$E(r_x) = c + \beta_x \left[E(r_y) - c \right]$$

where

$$\beta_x = \frac{Cov(x, y)}{\sigma_y^2}$$

Proposition: If there exists a risk-free asset with return r_f, then the standard security market line (SML) relationship holds:

$$E\left(r_x\right) = r_f + \beta_x \left[E\left(r_x\right) - r_f \right]$$

where

$$\beta_x = \frac{\operatorname{cov}(x, M)}{\sigma_M^2}$$